Algebraic geometry 2

Exercise Sheet 8

PD Dr. Maksim Zhykhovich Summer Semester 2026, 04.06.2026

Exercise 1. Let k be a field and let $X = \operatorname{Spec} A$ be an affine scheme of finite type over k. Let $x \in X$ and denote by $k(x) := \mathcal{O}_{X,x}/m_x$ the residue field of x. Show that $k(x)$ is algebraic over k if and only if x is a closed point of X (that is, x corresponds to a maximal ideal in A).
Hint: Let ρ be a prime ideal in A corresponding x. Recall that

$$k(x) \simeq A_\rho/\rho A_\rho \simeq \operatorname{Frac}(A/\rho)$$

and use Zariski's Lemma from Commutative Algebra: if a field K is a finitely generated k-algebra then K is algebraic over k.

Exercise 2. Let X be a scheme locally of finite type over a field k. Show that the set of closed points of X is dense.
Hint: Use Exercise 1.

Exercise 3. Let k be a field.
(1) Consider the ring of formal power series $k[[t]]$ and let $X = \operatorname{Spec} k[[t]]$ be an affine scheme over k. Describe the underlying topological space of X and show that the set of closed points is not dense in X.
Hint: Recall that $k[[t]]$ a a discrete valuation ring.
(2) Deduce from (1) and Exercise 2 that $k[[t]]$ is not finitely generated as k-algebra.

Exercise 4. Let B be a graded ring and $f \in B$ a homogeneous element of degree > 0.
(1) Show that the map

$$\theta : D_+(f) \to \operatorname{Spec} B_{(f)},\ \rho \mapsto \rho B_f \cap B_{(f)},$$

is bijective.
Hint: Show that the map $\psi : \operatorname{Spec} B_{(f)} \to D_+(f)$ constructed in the proof of Theorem 3.38 is the inverse of θ.
(2) Let $g \in B$ be a homogeneous element of degree > 0, such that $D_+(g) \subset D_+(f)$. Show that $\theta(D_+(g)) = D(h)$, where $h = g^{\deg f}/f^{\deg g} \in B_{(f)}$.